J. Fluid Mech. (2009), vol. 629, pp. 299–309. © 2009 Cambridge University Press
doi:10.1017/S002211200900679X Printed in the United Kingdom

Steady nonlinear diffusion-driven flow

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(Received 17 December 2008 and in revised form 22 February 2009)

An imposed normal temperature gradient on a sloping surface in a viscous stratified fluid can generate a slow steady flow along a thin 'buoyancy layer' against that surface, and in a contained fluid the associated mass flux leads to a broader-scale 'outer flow'. Previous analysis for small values of the Wunsch–Phillips parameter R is extended to the nonlinear case in a contained fluid, when the imposed temperature gradient is comparable with the background temperature gradient. As for the linear case, a compatibility condition relates the buoyancy-layer mass flux along each sloping boundary to the outer-flow temperature gradient. This condition allows the leading-order flow to be determined throughout the container for a variety of configurations.

1. Introduction

In concurrent and closely related studies, Wunsch (1970) and Phillips (1970) demonstrated that flow is generated if a boundary surface is sloping in an otherwise quiescent linearly stratified fluid. More recently, Page & Johnson (2008) extended that analysis to the case of a contained fluid when the imposed temperature gradient at the boundary is small relative to the background temperature gradient. Their governing equations are linear, which assists in the analysis of the flow structure, but they are not applicable to many practical situations. This paper extends their analysis to where the background temperature gradient is significantly affected by the motion so that the results can be compared directly with those in Woods (1991) and Quon (1989) and provide a foundation for determining the corresponding unsteady flow, as in the experiments of Peacock, Stocker & Aristoff (2004).

Wunsch (1970) and Phillips (1970) show that the key parameter for these flows is $R = \sqrt{\nu^*\kappa^*}/N^*L^{*2}$, in terms of the kinematic viscosity ν^*, thermal diffusivity κ^*, the buoyancy frequency N^* and a typical length scale L^*. As in most previous studies, it is assumed here that $R \ll 1$, and under those conditions there are three main regions of the flow: the so-called buoyancy layer, originally described by Wunsch (1970) and Phillips (1970); an 'outer flow' which occupies the bulk of the container; and horizontal '$R^{1/3}$ layers' which connect the mass flux between those two regions in some circumstances. (Here the direction opposite to the gravitational force is referred to as 'vertical'.)

The scaling and governing equations for this problem are outlined in § 2. The three key regions of the flow are described in § 3, leading to an ordinary differential equation

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which governs the ‘outer flow’. The analysis is illustrated in § 4 for the flow in a tilted square container, in § 5 for a circular container and in § 6 for a semi-infinite fluid.

2. Configuration and governing equations

The steady two-dimensional flow of a viscous stratified fluid is considered in a closed container of typical length scale L^*. A steady temperature variation is maintained in the fluid by specifying the normal temperature gradient T_n^* on the container walls with a zero overall total heat transfer, and the resulting temperature gradient variations within the container drive a steady broadscale interior motion (called the ‘outer flow’ here). In this paper the temperature is not assumed to be dominated by a constant vertical gradient, and so the governing equations differ from those of Wunsch (1970) and Phillips (1970).

A Cartesian coordinate system (x^*, z^*) is defined with gravitational acceleration g^* in the negative z^*-direction and corresponding velocity components denoted as (u^*, w^*). The temperature is denoted by $T^*(x^*, z^*)$, and the Boussinesq approximation is used, based upon a constant background density ρ_{00}^*. The coefficient of thermal expansion α^*, thermal diffusivity and kinematic viscosity are all taken to be constant.

To obtain the non-dimensional governing equations, lengths are scaled using L^*, so $(x, z) = (x^*, z^*)/L^*$, and an appropriate temperature scale ΔT^* is chosen, for example, L^* times the magnitude of variations in the imposed temperature gradient around the boundary. A non-dimensional temperature $T(x, z)$ is then defined through

$$T^*(x^*, z^*) = T_{00}^* + (\Delta T^*)T(x, z), \tag{2.1}$$

where T_{00}^* is the ‘average’ background temperature. (In contrast, Page & Johnson (2008) used the constant background temperature gradient $\mathrm{d}T_0^*/\mathrm{d}z^*$ to determine the temperature scale, so their scaled linearized temperature, written as $\tilde{T}$ here, is equivalent to $T = 4z + 2\epsilon\sqrt{\sigma}\ \tilde{T}$.)

The velocity components of the flow are non-dimensionalized with L^* and the buoyancy frequency $N^* = (g^*\alpha^*\Delta T^*/L^*)^{1/2}$, so $(u, w) = (u^*, w^*)/N^*L^*$, without the factor of ϵ used in Page & Johnson (2008). The pressure in the fluid is predominantly hydrostatic, and variations are quantified by a scaled pressure p that is non-dimensionalized with $\rho_{00}^*(N^*L^*)^2$.

As noted by Wunsch (1970), the key dynamical parameter in this problem is $R = \sqrt{\nu^*\kappa^*}/N^*L^{*2}$. The governing equations for steady flow can then be written as

$$uu_x + wu_z = -p_x + R\sqrt{\sigma}\ \nabla^2 u, \tag{2.2}$$

$$uw_x + ww_z = -p_z + T + R\sqrt{\sigma}\ \nabla^2 w, \tag{2.3}$$

$$uT_x + wT_z = (R/\sqrt{\sigma}\ )\nabla^2 T, \tag{2.4}$$

where $\sigma = \nu^*/\kappa^*$ is the Prandtl number. The continuity equation $u_x + w_z = 0$ allows a streamfunction ψ to be defined with

$$u = \partial\psi/\partial z \quad \text{and} \quad w = -\partial\psi/\partial x. \tag{2.5}$$

Taking into account the different scaling of the pressure and temperature here, without the factors of $\sqrt{\sigma}$ used in Page & Johnson (2008) and Wunsch (1970), the nonlinear equations (2.2)–(2.4) are equivalent to (1)–(3) in Wunsch (1970) when his $\epsilon = 1$. It is assumed that $R \ll 1$, with σ taken to be $O(1)$ with respect to R, and under those conditions it will be seen that p and T are dominantly functions of z over most of the container. The only significant term on the left-hand sides of (2.2)–(2.4) turns out to be wT_z, and the z-variation of the coefficient of w in that term provides the key difference from the linear analysis in Page & Johnson (2008).

As in Page & Johnson (2008), the value of $T_n = \partial T/\partial n$ is specified around the boundary, where n is the outward normal, in order to provide the 'driving force' for the steady flow. Non-slip boundary conditions are used, so both components of the velocity (u, w) vanish at the container walls. The (arbitrary) zero for the temperature is chosen so that the integral around the boundary of T is zero, and the constancy of the total heat in the container requires that the integral of T_n around the boundary must vanish.

3. Flow regions

As in Page & Johnson (2008), there are three key flow regions for steady 'diffusion-driven flow' in a closed container: on vertical or sloping surfaces there are 'buoyancy layers'; on some horizontal lines there can be thin '$R^{1/3}$ layers'; while the remainder is the 'outer flow'. Modifications to each of these regions in the nonlinear case are described below.

3.1. *The buoyancy layer*

This is arguably the most important region, as it provides the driving force behind the flow, via the specified value of T_n on a sloping boundary. It has thickness $O(R^{1/2})$ and can form on any surface that is not horizontal. In the nonlinear case its thickness can vary along the surface.

The flow above a sloping plane surface at angle α anticlockwise from horizontal is considered initially, although the results can be extended to surfaces with varying slope. In a rotated coordinate system $(\hat{x}, \hat{z})$, with $\hat{x} = x\cos\alpha + z\sin\alpha$, $\hat{z} = -x\sin\alpha + z\cos\alpha$ and velocity components $(\hat{u}, \hat{w})$, the steady equations of motion (2.2)–(2.4) become

$$\hat{u}\hat{u}_{\hat{x}} + \hat{w}\hat{u}_{\hat{z}} = -p_{\hat{x}} + T\sin\alpha + R\sqrt{\sigma}\,\hat{\nabla}^2\hat{u}, \tag{3.1}$$

$$\hat{u}\hat{w}_{\hat{x}} + \hat{w}\hat{w}_{\hat{z}} = -p_{\hat{z}} + T\cos\alpha + R\sqrt{\sigma}\,\hat{\nabla}^2\hat{w}, \tag{3.2}$$

$$\hat{u}T_{\hat{x}} + \hat{w}T_{\hat{z}} = (R/\sqrt{\sigma}\,)\hat{\nabla}^2 T, \tag{3.3}$$

with $\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0$. At $\hat{z} = 0$ the boundary conditions $\hat{u} = \hat{w} = 0$ are applied, with a specified value of $\partial T/\partial\hat{z}(\hat{x}, 0) = -T_n(\hat{x})$ for a given function T_n.

For $R \ll 1$ and $\alpha \neq 0$ an expansion of the solution is sought of the form

$$p = \hat{p}_0(\hat{x}, \zeta) + R^{1/2}\hat{p}_1(\hat{x}, \zeta) + \dots, \quad \hat{u} = R^{1/2}\hat{u}_1(\hat{x}, \zeta) + R\hat{u}_2(\hat{x}, \zeta) + \dots, \tag{3.4a}$$

$$T = \hat{T}_0(\hat{x}, \zeta) + R^{1/2}\hat{T}_1(\hat{x}, \zeta) + \dots, \quad \hat{w} = R\hat{w}_2(\hat{x}, \zeta) + R^{3/2}\hat{w}_3(\hat{x}, \zeta) + \dots, \tag{3.4b}$$

using the boundary-layer coordinate $\zeta = \hat{z}/R^{1/2}$. A scaled streamfunction $\hat{\psi}$ is introduced such that $\hat{u} = \partial\hat{\psi}/\partial\hat{z}$, with $\hat{\psi} = 0$ at $\hat{z} = 0$, and expanded in the form $\hat{\psi} = R\hat{\psi}_2(\hat{x}, \zeta) + O(R^{3/2})$ so that, for example, $\hat{u}_1 = \partial\hat{\psi}_2/\partial\zeta$.

The leading-order terms in the equations imply that both $\hat{p}_0$ and $\hat{T}_0$ are functions of $\hat{x}$ only and are determined by matching with the outer flow for large ζ. Integration of the $O(1)$ terms in (3.2) implies that $\hat{p}_1(\hat{x}, \zeta) = \zeta\,\hat{T}_0(\hat{x})\cos\alpha + \hat{p}_{10}(\hat{x})$, where $\hat{p}_{10}(\hat{x})$ is determined by matching, while the $O(R^{1/2})$ terms in (3.1) and (3.3) give that

$$0 = -\frac{\partial\hat{p}_1}{\partial\hat{x}} + \hat{T}_1\sin\alpha + \sqrt{\sigma}\,\frac{\partial^2\hat{u}_1}{\partial\zeta^2} \quad \text{and} \quad \hat{u}_1\hat{T}_0'(\hat{x}) = \frac{1}{\sqrt{\sigma}}\frac{\partial^2\hat{T}_1}{\partial\zeta^2}. \tag{3.5}$$

Note that the nonlinear term on the left-hand side of (3.1) is $O(R)$, so it does not affect the expansion at this order, but the $O(R^{1/2})$ term on the left-hand side of (3.3) introduces a parametric $\hat{x}$-dependence through $\hat{T}_0'$.

Solutions of (3.5) are sought such that $\hat{u}_1 = 0$ on $\zeta = 0$, with no exponentially growing terms for large ζ, and these have the form

$$\hat{u}_1(\hat{x}, \zeta) = \hat{U}_1(\hat{x}) \exp(-\beta\zeta) \sin(\beta\zeta), \tag{3.6}$$

$$\hat{T}_1(\hat{x}, \zeta) = 2\operatorname{cosec}\alpha \sqrt{\sigma}\ \beta^2 \hat{U}_1(\hat{x}) \exp(-\beta\zeta) \cos(\beta\zeta) + \operatorname{cosec}\alpha \frac{\partial \hat{p}_1}{\partial \hat{x}}(\hat{x}, \zeta), \tag{3.7}$$

where $\beta = [(1/4)\sin\alpha\ \hat{T}_0'(\hat{x})]^{1/4} > 0$. The key difference between these solutions and those in (3.5) and (3.6) of Page & Johnson (2008) is the $\hat{x}$-dependence of the layer thickness $R^{1/2}/\beta$, as β depends on the temperature gradient in the outer flow. In (3.6), $\hat{U}_1(\hat{x})$ can be determined from the boundary condition on the temperature $\partial\hat{T}_1/\partial\zeta(\hat{x}, 0) = -T_n(\hat{x})$, and hence the scaled $O(R)$ mass flux $\hat{\psi}_2(\hat{x}, \infty) = (1/2)\hat{U}_1(\hat{x})/\beta$ satisfies

$$[\sqrt{\sigma}\ \hat{\psi}_2(\hat{x}, \infty) - \cot\alpha]\ \hat{T}_0'(\hat{x}) = T_n(\hat{x}). \tag{3.8}$$

This is the nonlinear version of the compatibility condition (3.7) in Page & Johnson (2008), connecting the scaled mass flux and temperature gradient at the outer edge of the buoyancy layer to the imposed value of T_n without explicitly determining the flow in that layer.

Section 3.2 shows that the temperature T in the outer flow is independent of x to leading order, and so $\hat{T}_0'(\hat{x}) = \sin\alpha T_0'(z)$, where $T_0(z)$ is the leading-order outer-flow temperature at height z. In terms of the original variables (x, z), it follows from (3.8) that ψ and T at the outer edge of the buoyancy layer are related through

$$\psi = \frac{R}{\sqrt{\sigma}\ \sin\alpha}\left[\cos\alpha + \frac{T_n}{\partial T/\partial z}\right] + O\left(R^{3/2}\right). \tag{3.9}$$

On parts of the boundary where $T_n = 0$ (the insulating condition of zero heat flux), (3.9) gives $\psi = R\cot\alpha/\sqrt{\sigma}$, as noted by Wunsch (1970) and Phillips (1970) (and used by Woods 1991). Also, when (3.8) and (3.9) are applied to $T = 4z + 2\epsilon\sqrt{\sigma}\ \tilde{T}$ for $\epsilon \ll 1$ and linearized about the background temperature gradient, they yield the equivalent compatibility relations (3.7) and (3.8) in Page & Johnson (2008) in terms of their linearized variables $\tilde{T}$ and $\tilde{\psi} = \psi/\epsilon$.

As noted in Page & Johnson (2008) and also for the particular case of insulating ($T_n = 0$) boundaries by Woods (1991), this analysis remains valid when α varies on an $O(1)$ length scale, so it is not limited to containers with uniformly sloping boundaries.

3.2. *The outer flow*

For the linear case it was noted that an $O(R)$ mass flux can be generated at the outer edge of the buoyancy layer when the right-hand side of (3.9) varies with $\hat{x}$. In the 'outer-flow' region, for which (x, z) is $O(1)$, this induces $O(R)$ velocities (u, w) and $O(1)$ temperature perturbations T. To leading order, (2.2)–(2.4) therefore become

$$0 = -p_x, \quad 0 = -p_z + T \quad \text{and} \quad uT_x + wT_z = (R/\sqrt{\sigma}\ )\nabla^2 T. \tag{3.10}$$

As in Page & Johnson (2008), the solution can be expanded in powers of $R^{1/2}$ as

$$p = p_0(x, z) + R^{1/2} p_1(x, z) + \ldots, \qquad T = T_0(x, z) + R^{1/2} T_1(x, z) + \ldots, \tag{3.11a}$$

$$w = R w_2(x, z) + R^{3/2} w_3(x, z) + \ldots, \quad \psi = R\psi_2(x, z) + R^{3/2}\psi_3(x, z) + \ldots, \tag{3.11b}$$

where ψ is given by (2.5). From the first two equations of (3.10), both p_0 and T_0 are independent of x, and so the leading-order solution can be written in terms of two functions $f(z)$ and $g(z)$ with

$$T_0 = f(z), \quad w_2 = \frac{f''(z)}{f'(z)\sqrt{\sigma}} \quad \text{and} \quad \psi_2 = -\frac{xf''(z)}{f'(z)\sqrt{\sigma}} + g(z). \tag{3.12}$$

For a flow in a closed container, constraints are imposed on the functions f and g by the conditions (3.9) at the two boundaries in x, and, as for the linear case, this determines both f' and g to within an arbitrary constant.

Applying the analysis above to a region $x_- < x < x_+$ which has boundaries at $x_\pm$ with slope $\alpha_\pm$, where $\cot\alpha_\pm = x'_\pm(z)$, gives the $O(R)$ mass flux at each boundary as

$$\psi_2(x_\pm, z) = \frac{\cot\alpha_\pm}{\sqrt{\sigma}} + \operatorname{cosec}\alpha_\pm \frac{T_{n\pm}}{\sqrt{\sigma}\ f'(z)} \tag{3.13}$$

from (3.9). In terms of the container width $L(z) = [x_+(z) - x_-(z)]$, it follows that

$$w_2 = -[\psi_2(x_+, z) - \psi_2(x_-, z)]/L(z) = -\frac{L'(z)}{\sqrt{\sigma}\ L(z)} - \frac{[\operatorname{cosec}\alpha\ T_n]_-^+}{\sqrt{\sigma}\ L(z) f'(z)}, \tag{3.14}$$

and equating this with the second equation in (3.12) requires that f' must satisfy the differential equation

$$(Lf')' + [\operatorname{cosec}\alpha\ T_n]_-^+ = 0, \tag{3.15}$$

in terms of the given boundary condition T_n and slope $\alpha_\pm$ at $x_\pm$. Once f' has been determined, g can be found by using the third equation in (3.12) at either x_- or x_+ with (3.13). The unique specification of f, and hence the 'outer flow', is completed by noting that the two unknown constants in the solution of (3.15) must satisfy the requirement that the average values of both T and $\partial T/\partial n$ around the boundary are zero (see § 2).

When $T_n = 0$ at both boundaries (3.15) implies that Lf' is independent of z, equivalent to the observation by Woods (1991) that $A\rho_z$ is constant for a region of width $A(z)$.

A significant feature of the outer-flow solution (3.12) is that the strength of the recirculation, as measured by $\max|\psi_2|$ say, is proportional to f''/f' rather than the absolute magnitude of the outer-flow temperature gradient, $\max|f'|$ say. This is because the outer-flow motion is driven by the mass flux along the buoyancy layers, which does not depend explicitly on the temperature gradient. For example, where $T_n = 0$ on a boundary the induced mass flux is proportional to $\kappa^* \cot\alpha$ in dimensional terms (Woods 1991) which is independent of $\Delta T^*/L^*$, provided that the *variation* of the temperature gradient around the boundary is much larger than $\nu^*\kappa^*/g^*\alpha^* L^{*4}$. It is the relative variation of the temperature gradient that determines the strength of the recirculation rather than its absolute size or indeed the proportion of the boundary over which it is non-zero.

3.3. *The $R^{1/3}$ layer*

Page & Johnson (2008) showed that thin horizontal $R^{1/3}$ layers, equivalent to Stewartson $E^{1/3}$ layers, occur in the flow under some circumstances. For example, when there is a fluid source (or sink) at a point z_0 on a sloping (or vertical) boundary an $R^{1/3}$ layer centred on $z = z_0$ redistributes fluid across the container. Page & Johnson (2008) also noted that an analysis similar to Moore & Saffman (1969) implies that

$$\text{both } T \text{ and } \partial T/\partial z \text{ must be continuous across any } R^{1/3} \text{ layer,} \tag{3.16}$$

so that both f and f' are continuous at $z = z_0$ (but that $w \propto f''$ may be discontinuous). This conclusion remains valid for the nonlinear case examined here, although the scale thickness of the layer is $(RL/\sqrt{f'(z_0)})^{1/3}$, where L is the length of the container, so that the layer broadens as the local temperature gradient decreases (personal observation).

In combination with the analysis in § 3.2, the condition (3.16) determines unique solutions for f and g, and hence the overall outer flow, when $R^{1/3}$ layers are present.

4. Flow in a tilted square container

To compare the nonlinear results directly with the linear theory, the flow in a square container that has been tilted by 45° is considered. The square is taken to have sides of length $\sqrt{2}$ so that the boundaries are at $z = 1 \pm (1 - |x|)$ for $|x| \leqslant 1$.

As in Page & Johnson (2008), a non-zero steady flow is forced by using a piecewise constant boundary condition $\partial T/\partial n = T_n$ around the container, with discontinuities at $z = 1/2$, 1 and 3/2. These conditions ensure antisymmetry of T and ψ about $z = 1$ with $T_n = -2\sqrt{2}$ for $z < 1/2$, $T_n = -2\sqrt{2} + 2\epsilon\sqrt{\sigma}$ for $1/2 < z < 1$, $T_n = 2\sqrt{2} - 2\epsilon\sqrt{\sigma}$ for $1 < z < 3/2$ and $T_n = 2\sqrt{2}$ for $z > 3/2$. For $\epsilon \ll 1$ this problem corresponds to that considered by Page & Johnson (2008), while for $\epsilon = \sqrt{2/\sigma}$ it gives that $T_n = 0$ for $1/2 < z < 3/2$, as assumed by Woods (1991). Only the solution for $z < 1$ is described below, as that for $z > 1$ follows by replacing z with $(2 - z)$ and changing the signs of T and ψ.

Using the notation of § 3.2, for $z < 1$ this configuration has $x_\pm = \pm z$, $\alpha_\pm = \pm(1/4)\pi$ and $L = 2z$. For $z < 1/2$, where $T_n = -2\sqrt{2}$, (3.15) gives that

$$(2zf')' + [\sqrt{2} - (-\sqrt{2})](-2\sqrt{2}) = 0 \quad \text{so that} \quad f'(z) = 4 + c_1/z, \tag{4.1}$$

where c_1 must be zero to avoid a singularity at $z = 0$ (as in Page & Johnson (2008)), and so $f'(z) = 4$. Since $g(z) = 0$, from symmetry about $x = 0$, it follows that $\psi_2 = 0$ for $z < 1/2$, and there is no outer flow. Hence $T_0(z) = 4(z - 1/2) + T_0(1/2)$, where $T_0(1/2)$ is determined below.

For $1/2 < z < 1$ a similar analysis gives that $f'(z) = 4 - 2\epsilon\sqrt{2\sigma} + c_2/z$, and this satisfies the condition (3.16) across the $R^{1/3}$ layer at $z = 1/2$ when $c_2 = \epsilon\sqrt{2\sigma}$. Since $T_0(1) = 0$ by symmetry, it follows that $T_0(z) = (4 - 2\epsilon\sqrt{2\sigma})(z - 1) + \epsilon\sqrt{2\sigma}\,\ln z$ and that

$$\psi_2(x, z) = \frac{\epsilon\sqrt{2}\,(x/z)}{4z + \epsilon\sqrt{2\sigma}\,(1 - 2z)} \quad \text{for} \quad \tfrac{1}{2} < z < 1. \tag{4.2}$$

For $\epsilon \ll 1$ the $O(\epsilon)$ term in (4.2) matches (4.7) of Page & Johnson (2008), noting that $T = 4z + 2\epsilon\sqrt{\sigma}\,\tilde{T}$ and $\psi = \epsilon\tilde{\psi}$ here. This implies that $w_2 \propto 1/z^2$ for $1/2 < z < 1$, and there is a varying *detrainment* from the buoyancy layer over that range (inadvertently described as entrainment in Page & Johnson (2008)). The outer-flow streamlines are shown in figure 1(c) of Page & Johnson (2008).

For $0 < \epsilon < \sqrt{2/\sigma}$, $T_n < 0$ for $1/2 < z < 1$, and there is also a mass flux out of the buoyancy layer, as given by (3.9). The vertical velocity of the outer flow is $w_2(z) = -\epsilon\sqrt{2}/(4z^2 + \epsilon\sqrt{2\sigma}\,z(1 - 2z))$, which is negative and increases in magnitude as z decreases. This is due to the combined effect of the buoyancy-layer efflux and the narrowing container width. Streamlines of the outer flow for the typical case $\epsilon = (1/2)\sqrt{2/\sigma}$ are shown in figure 1(a), and numerical solutions of the full equations (2.2)–(2.4) for $R = 0.0001$ and $\sigma = 1$ are shown in figure 1(c), obtained by using a finite-difference method on a uniform 300 × 300 grid. As in Page & Johnson (2008), these

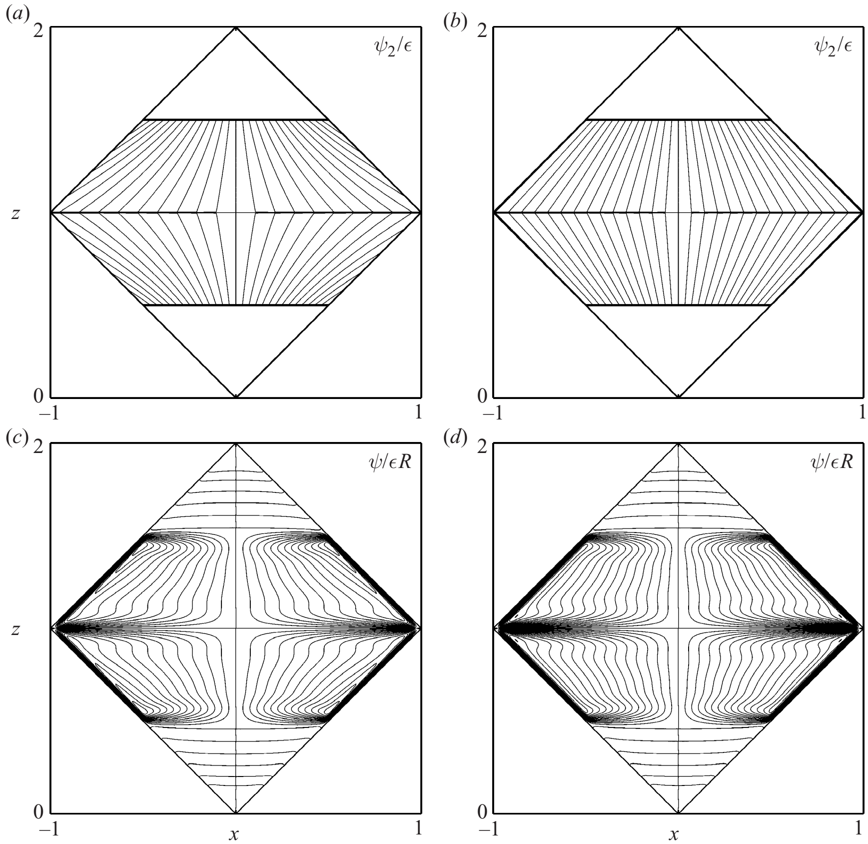

FIGURE 1. Streamlines in a tilted square container based on the outer-flow solution ψ_2/ϵ for the boundary conditions in §4 when (a) $\epsilon=(1/2)\sqrt{2/\sigma}$ and (b) $\epsilon=\sqrt{2/\sigma}$, using $\Delta\psi_2=0.05/\sqrt{\sigma}$. Also shown are equivalent numerical solutions $\psi/\epsilon R$ of the full equations for $R=0.0001$ and $\sigma=1$ when (c) $\epsilon=(1/2)\sqrt{2}$ and (d) $\epsilon=\sqrt{2}$.

plots show similar features, affirming the applicability of the asymptotic solutions for the outer flow when $R \ll 1$. Although not shown, the outer-flow temperature perturbations from the linear background gradient $4(z-1)$ are similar in character to those of $\tilde{T}$ when $\epsilon \ll 1$ (see figure 1b of Page & Johnson (2008)).

For $\epsilon=\sqrt{2/\sigma}$, which corresponds to applying $T_n=0$ for $1/2<z<1$, the denominator of (4.2) is equal to 2 and $\psi_2(x,z)=x/(\sqrt{\sigma}\ z)$. This is the same form of steady solution proposed by Woods (1991) for the $T_n=0$ case, and it implies that $w_2 \propto 1/z$, as in his (2.3). Figure 1(b) shows that the outer-flow streamlines immediately outside the boundary layer are parallel to the container boundaries, in contrast to those in figure 1(a). Numerical solutions for $R=0.0001$ are also shown in figure 1(d), and their features are broadly consistent with those properties.

As noted above, the buoyancy layers start with non-zero mass flux from the $R^{1/3}$ layer at $z=1/2$ and (for $\epsilon<\sqrt{2/\sigma}$ at least) lose fluid as they move up the sloping surfaces. Antisymmetry requires that $\psi=0$ at $z=1$, and so the remaining fluid is expelled over a short distance as $z\rightarrow 1$ and recirculated along another $R^{1/3}$ layer near $z=1$. There are also adjustment regions of size $R^{1/2}\times R^{1/2}$ in the corners, where the two buoyancy layers meet, but they are localized. These features are apparent in the $R = 0.0001$ numerical solutions in figure 1(c, d), and as R is decreased the layers reduce in thickness, and the features of the outer flow are resolved more accurately, including the correspondence between T and f.

Despite its benign features, with no flow and a constant temperature gradient, the solution for $0 < z < 1/2$ provides the driving force for the steady recirculating motion within the container, as it supplies a source of energy to the flow for $1/2 < z < 1$ through a steady vertical temperature gradient.

5. Flow in a circular container

The effect of boundaries with varying slope can be examined by considering a circular container $x^2 + (z-1)^2 < 1$, similar to that considered by Quon (1989). The temperature gradient boundary conditions considered here are that $T_n = 0$ on $r = 1$ for $a < z < (2-a)$ for $a > 0$ with $T_n = \sin\theta$ otherwise, where (r, θ) are polar coordinates about $(x, z) = (0, 1)$. In effect, this extends the tilted-square problem to a circular container for the case $\epsilon = \sqrt{2/\sigma}$, so some of the features of that solution should be similar to those described in §4 – including the antisymmetry about $z = 1$, the driven flow over $a < z < (2-a)$ and the $R^{1/3}$ layers centred on $z = a$ and $z = (2-a)$.

For $z < 1$ the container has $x_\pm = \pm\sqrt{2z - z^2}$ and $\alpha_\pm = \pm\arccos(1-z)$ so that $L = 2\sqrt{2z - z^2}$. When $a < z < 1$ analysis similar to that in §4 for $T_n = 0$ gives that $f'(z)\sqrt{2z - z^2}$ is constant, and so $f(z) = -d_2 \arcsin(1-z)$ for some constant d_2, using that $T_0(1) = 0$ by antisymmetry. This solution is equivalent to (2.15) in Woods (1991), and for any d_2 it corresponds to a streamfunction of the form

$$\psi_2(x, z) = \frac{x(1-z)}{\sqrt{\sigma}\; z(2-z)} \quad \text{for} \quad a < z < 1. \tag{5.1}$$

The flow has $w_2 < 0$, and fluid is detrained from the buoyancy layer as z increases. Unlike in §4, however, the buoyancy layer has emptied once it reaches $z = 1$, and so there is no forcing for an $R^{1/3}$ layer centred on that level.

For $z < a$, where $T_n = \sin\theta = (z-1)$, (3.15) gives

$$(2\sqrt{2z - z^2} f')' + 2(z-1)/\sqrt{2z - z^2} = 0, \quad \text{so} \quad f'(z) = 1 + d_1/\sqrt{2z - z^2}, \tag{5.2}$$

where $d_1 = 0$ to avoid a singularity at $z = 0$. Therefore $f'(z) = 1$ for $0 < z < a$, and also $g(z) = 0$ by symmetry about $x = 0$, so $\psi_2 = 0$, and there is no outer flow in this region (irrespective of the value of $a > 0$). The corresponding temperature is $T_0(z) = z - a - d_2 \arcsin(1-a)$, where $d_2 = \sqrt{2a - a^2}$ from applying (3.16) at $z = a$.

The streamlines for the outer flow (5.1) for this geometry are shown in figure 2(a) for $a = 0.1$. As noted earlier, the strength of this outer-flow recirculation is independent of the temperature gradient and in particular the value of d_2 over $a < z < (2-a)$. It is also independent of a, so as $a \to 0$ the strength of the recirculating flow is unaffected until the assumptions are no longer valid, which occurs once the buoyancy layer has the same thickness as the $R^{1/3}$ layer or when $a = O(\sin^2\alpha) = O(R^{2/3})$.

Quon (1989) considered this same problem with $T_n = 0$ on all of the boundary but assumed that the temperature gradient was constant throughout the outer flow. He deduced that ψ must be a function of z only – in contrast to the form of the third equation in (3.12), which depends upon both x and z. This led to a recirculating flow illustrated in his figure 4(a), with thin layers near $x = 0$, and he proposed that these layers arise when a ‘forced axial flow [is] supplying heat at the top and draining off heat at the bottom of the cylinder’ (Quon 1989, p. 202). However, no such layers were

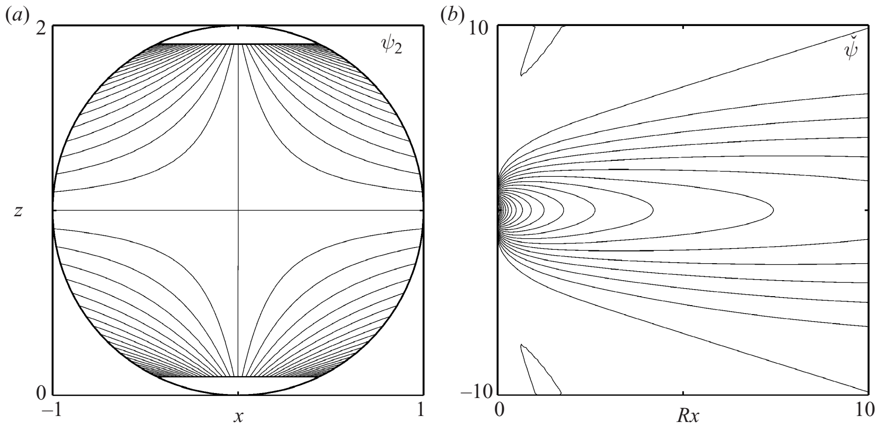

FIGURE 2. (*a*) Streamlines for the outer flow ψ_2 in a circular container with the boundary conditions specified in § 5 when $a = 0.1$, using $\Delta\psi_2 = 0.05/\sqrt{\sigma}$. (*b*) The scaled streamfunction $\check{\psi}$ in a wide container when $x = O(R^{-1})$, using $\Delta\check{\psi} = 0.05 \cot\alpha/\sqrt{\sigma}$.

present for the similar type of forced steady flow in Page & Johnson (2008), nor are they necessary for the equivalent nonlinear solution described in § 4 here.

In contrast, the outer-flow analysis above, along with a buoyancy layer near $r = 1$ and $R^{1/3}$ layers near $z = a$ and $z = (2 - a)$, presents a simple and self-consistent asymptotic solution for the same problem. The corresponding temperature profile $T_0(z)$ for $a < z < (2 - a)$ is also in accord with the analysis by Woods (1991) that LT_z is constant when $T_n = 0$, which in turn leads to his (2.15). As noted at the end of § 4, the recirculation (5.1) is generated by the imposed temperature gradient for the 'quiescent' region for $z < a$, where $\psi_2 = 0$ and T_0 is linear in z, and this represents the same kind of 'forced axial flow' envisaged by Quon (1989) – especially when $a \ll 1$.

6. Flow in a wide container

The examples in § 4 and § 5 illustrate the form of the outer flow in a container of width $O(1)$, but it is the mass-flux variations in the z-direction, driven by the buoyancy layer, which force the fluid to recirculate. Recirculation also occurs in a semi-infinite fluid with only one sloping wall, provided T_n varies along the boundary. As an example, consider a semi-infinite fluid to the right of a sloping plane at $z = x \tan\alpha$, where $\alpha < 0$ is $O(1)$, similar to the configuration in Wunsch (1970) and Phillips (1970) but with $T_n = -\cos\alpha$ for $|z| > 1$ and $T_n = 0$ for $|z| < 1$. Fluid travels up the buoyancy layer from $z = -1$ towards $z = 1$ and recirculates in an 'outer flow' to close the mass flux.

Following § 3.2 when $L \gg 1$, (3.15) implies that $f'' = 0$ in the outer flow and that f' is constant everywhere. For $|z| > 1$ the outer solution $T_0' = 1$, and $\psi_2 = 0$ for all $x > z \cot\alpha$ satisfies the boundary condition $T_n = -\cos\alpha$ without a buoyancy layer. When $|z| < 1$, however, the outer flow has T_0' constant, so a buoyancy layer is required near $x = z \cot\alpha$, and from (3.9) it must carry a mass flux of $\psi_2(z \cot\alpha, z) = \cot\alpha/\sqrt{\sigma}$. From the third equation in (3.12), it follows that $\psi_2(x, z) = g(z) = \cot\alpha/\sqrt{\sigma}$ everywhere in the outer flow when $|z| < 1$. The fluid ejected from the buoyancy layer at $z = 1$ therefore moves out along an $R^{1/3}$ layer and then back along a similar layer centred on $z = -1$. Both T_0 and T_0' must be continuous across those layers, from (3.16), and so

$T_0 = z$ everywhere in the outer flow. This remains valid under the conditions assumed in § 3.2, where $x = O(1)$, and on that scale the $R^{1/3}$ layers both extend to infinity in x.

A sink flow with an $R^{1/3}$ layer near $z = 0$ in a semi-infinite fluid is described by Koh (1966) with the streamfunction expressed in terms of a similarity variable $z/(Rx)^{1/3}$. For $x = O(1)$ the layer has width $O(R^{1/3})$, but it broadens to order-one values of z and thereby affects the outer-flow region, once x is $O(R^{-1})$. This problem is also similar to a body moving along the axis of an unbounded viscous rotating fluid, described by Hocking, Moore & Walton (1979), who observed a similar long x-scale. In terms of $X = Rx$ and using scaled variables $\psi = R\check{\psi}(X, z)$ and $T = z + R\check{T}(X, z)$ it follows from (2.2)–(2.4) that $\check{\psi}_X = -\check{T}_{zz}/\sqrt{\sigma}$ and $\check{T}_X = \sqrt{\sigma}\ \ \check{\psi}_{zzzz}$. The relevant solution for the case here is a linear superposition of the flow from a source at $(X, z) = (0, 1)$ and sink at $(0, -1)$, with

$$\check{\psi}(X, z) = (\cot\alpha/\sqrt{\sigma}\ )\ \left[f_0\big((z-1)/X^{1/3}\big) - f_0\big((z+1)/X^{1/3}\big)\right] \qquad (6.1)$$

in terms of the similarity solution f_0 described in § 1.4 of Koh (1966). The streamlines for this flow are shown in figure 2(*b*), indicating how the mass recirculates over this longer $x = O(R^{-1})$ scale for $R \ll 1$.

The analysis above implicitly assumes that $\alpha = O(1)$. For small α the mass flux along sloping boundary becomes large, and the buoyancy layer thickens to $O(\sqrt{R/\sin\alpha})$. In particular, for $\alpha = O(R^{1/3})$ the buoyancy layer and $R^{1/3}$ layers merge near the mass source and sink at $z = \pm 1$, and the solution above no longer applies.

7. Conclusions

This paper extends the flow structure for the steady recirculation of a contained flow in Page & Johnson (2008) to the nonlinear case, where the background temperature gradient is significantly affected by the motion. This allows a more complete analysis of the closed-container version of the problem considered by Wunsch (1970) and Phillips (1970). The buoyancy layer has non-constant thickness, but the overall features of the flow field remain similar to those for the linear case consider by Page & Johnson (2008), including the derivation of a ‘compatibility condition’ which relates the outer flow directly to the imposed normal temperature gradient on the boundary. Based on that condition, the outer flow is determined by solving a second-order ordinary differential equation (3.15). In particular, when the imposed normal temperature gradient T_n is zero on both horizontal boundaries over some range of values of z, LT_z is constant throughout the outer flow (where $L(z)$ is the container width), as also noted by Woods (1991). This analysis also leads to an alternative form of asymptotic solution from that proposed by Quon (1989) for flow in a circular container.

The approach used in this paper can be extended to the case of unsteady flow and in particular to a typical experimental situation in which the fluid is initially stagnant with a constant temperature gradient throughout. Three time scales can be identified for that problem, with the buoyancy and $R^{1/3}$ layers established, quickly followed by a more gradual decay to a stagnant fluid at a constant temperature.

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